We may observe that every algebraic identity, in which each term is of the second degree in so far as certain letters are concerned, may be given a geometric interpretation if each of such letters be used to represent a certain vector, and if the scalar but not the vector portion of the product be employed in the interpretation. That this is true follows from the fact that the non-commutative character of vector multiplication does not alter or affect the scalar portion of the product, if each term of such product contains either the product of two separate vectors or the square of some one vector; *i. e.*, if no term in the expanded form is of a degree higher or lower than the second in the letters used to designate vectors.

A solution similar to the foregoing may be employed in problem 377.

384. Proposed by S. LEFSEHETZ, Clark University.

Let ABC be a triangle, O a circle tangent to its three sides, T a variable tangent of O, which cuts the sides BC, CA, AB in a, b, c. Oa', Ob', Oc' the perpendiculars in O to Oa, Ob, Oc, cutting, respectively, T in points a', b', c'. Prove that Aa', Bb', Cc' meet in a point t, and find the locus of t when T varies. Purely geometrical proofs wanted.

No solution of this problem has been received.

385. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

Given a triangle ABC, find the radius of a circle touching two of its sides and a line parallel to the third, at a distance $d=u+2r$.

Solution by A. H. HOLMES, Brunswick, Maine.

Let a, b, and c be the sides of the given triangle, c the base. Then h=altitude of the triangle$=\dfrac{\sqrt{[4a^2c^2-(a^2-b^2+c^2)]}}{2c}$, and R=radius of the inscribed circle $=\dfrac{\sqrt{[4a^2c^2-(a^2-b^2+c^2)]}}{2(a+b+c)}$.

Put r=radius of circle touching a and b and a line parallel to c at a distance from c, $2r+u$. Then $h:R=h-(2r+u):r$.

$\therefore r=\dfrac{(h-u)R}{h+2R}$.

Putting for h and R their values in terms of a, b, and c, we have,

$$r=\frac{\sqrt{[4a^2c^2-(a^2-b^2+c^2)^2]}-u}{a+b+3c}.$$

CALCULUS.

306. Proposed by FRANCIS RUST, C. E., Pittsburg, Pa.

Express in elliptic integrals: $A_\theta=\displaystyle\int_0^\theta \frac{dx}{\sqrt{(1-x^4)}}$; $0<\theta<1$.

I **Solution by WALTER D. LAMBERT, A. M., University of Pennsylvania, Philadelphia, Pa.**

Correcting the inequality to read $0<\theta<1$, we find (by using the substitution $x=\cos\phi$, and by calling $\alpha=\cos^{-1}\theta$) that

$$A_\theta=\int_0^\theta \frac{dx}{\sqrt{[(1-x^2)(1+x^2)]}}=-\int_{\frac{1}{2}\pi}^{\alpha}\frac{\sin\phi\, d\phi}{\sqrt{[(1-\cos^2\phi)(1+\cos^2\phi)]}}$$

$$=\int_\alpha^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{[2-\sin^2\phi]}}=\frac{1}{\sqrt{2}}\int_\alpha^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{[1-\frac{1}{2}\sin^2\phi]}}$$

$$=\frac{1}{\sqrt{2}}\left[F\left(\frac{\pi}{2}\right)-F(\alpha)\right] \text{ modulus } \frac{1}{\sqrt{2}}$$

where $F\left(\frac{\pi}{2}\right)$ and $F(\alpha)$ are Legendre's elliptic integrals of the first kind.

As a numerical example take $\theta=\frac{1}{2}$. $\therefore \alpha=\frac{1}{3}\pi$.

$A_\theta=\frac{1}{\sqrt{2}}[1.8451-1.1424]=0.5032$, using the four-place "funktionentafeln" of Jahnke and Emde. As a verification, expand $\frac{1}{\sqrt{(1-x^4)}}$ by the binomial theorem and integrate:

$$A_\theta=\int_0^\theta\frac{dx}{\sqrt{[1-x^4]}}=\int_0^\theta(1+\tfrac{1}{2}x^4+\tfrac{3}{8}x^8+\tfrac{5}{16}x^{12}\ldots)dx=\theta+\frac{\theta^5}{10}+\frac{\theta^9}{24}+\tfrac{5}{208}\theta^{13}.$$

On substituting $\theta=\frac{1}{2}$ we get .5032 for A_θ as before.

II. **Solution by the PROPOSER.**

Let $x=\tan\omega$. Then $dx=\frac{d\omega}{\cos^2\omega}$. Substituting these values in the integral expression, we have

$$A_\theta=\int_0^\omega\frac{d\omega}{\cos^2\omega\sqrt{[(1-\tan^2\omega)(1+\tan^2\omega)]}}$$

$$=\int_0^\omega\frac{d\omega}{\sqrt{[\cos^2\omega-\sin^2\omega]}}=\int_0^\omega\frac{d\omega}{\sqrt{[1-2\sin^2\omega]}},$$

where $\tan\omega=\theta$ is used for the upper limit.

Now let $\sqrt{2}\sin\omega=\sin\phi$; then $1-2\sin^2\omega=\cos^2\phi$ and

$$d\omega=\frac{\cos\phi\, d\phi}{\sqrt{2}\sqrt{[1-\frac{1}{2}\sin^2\phi)]}}.$$

Whence $A_\theta = \frac{1}{2}\sqrt{2}\int_0^\phi \frac{d\omega}{\sqrt{[1-\frac{1}{2}\sin^2\phi]}}=\frac{1}{2}\sqrt{[2]}F(\frac{1}{2}\sqrt{2},\ \phi)$.

The amplitude ϕ is determined from $\theta=\tan\omega$, $\cos^2\omega=\frac{1}{1+\theta^2}$, $\sin^2\omega=\frac{\theta^2}{1+\theta^2}$, $\sin^2\phi=\frac{2\theta^2}{1+\theta^2}$, $\cos^2\phi=\frac{1-\theta^2}{1+\theta^2}$. Hence, $\tan^2\phi=\frac{2\theta^2}{1-\theta^2}$.

Referring to problem 303,

$$A=\int_0^1 \frac{dx}{\sqrt{[1-x^4]}}=\frac{\sqrt{\pi}}{4}\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})},$$

the well known formula $\Gamma(\theta)\Gamma(1-\theta)=\pi/\sin\pi\theta$, gives in our case $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})=\pi\sqrt{2}$. Whence, $\Gamma(\frac{3}{4})=\frac{\pi\sqrt{2}}{\Gamma(\frac{1}{4})}$, and therefore, $A=\frac{[\Gamma(\frac{1}{4})]^2}{4\sqrt{[2\pi]}}$.

This combined with the result in above, $A=\frac{1}{2}\sqrt{[2]}F^I\ (\frac{1}{2}\sqrt{2})$ yields $\Gamma(\frac{1}{4})=2\sqrt[4]{[\pi]}\{F^I\ (\frac{1}{2}\sqrt{2})\}^{\frac{1}{2}}=3.62561$, *0.5593811.

And similarly, $\Gamma(\frac{3}{4})=\frac{1}{2}\sqrt{[2]}\sqrt[4]{[\pi^3]}\{F^I\ (\frac{1}{2}\sqrt{2})\}^{-\frac{1}{2}}=1.225416$, *0.0882838.

These expressions for $\Gamma(\frac{1}{4})$ and $\Gamma(\frac{3}{4})$ in Legendre's F-functions are to my mind by far the most important consequences of evaluating integral A in gamma-functions. Without this evaluation $\Gamma(\frac{1}{4})$ and $\Gamma(\frac{3}{4})$ can be determined only by computing their natural logarithms by inconvenient series.

Also solved by V. M. Spunar, C. N. Schmall, and J. Scheffer.

MECHANICS.

253. Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Stroud, England.

R_1 and R_2 are ranges on a horizontal plane of particles projected with given velocity from A on the plane to pass through B. Show that $a(R_1+R_2)-R_1R_2=\frac{a^4}{c^2}$, where $c=AB$ and a is the horizontal projection of AB.

III. Solution by the PROPOSER.

If α, α_1 be angles of projection and β the angle AB makes with the horizontal, and v the velocity of projection, then $\alpha_1=\frac{1}{2}\pi-(\alpha-\beta)$.

And $\cos\beta=a/c$; $R_1.g=a^2\sin2\alpha$; $R_2.g=a^2\sin2(\alpha-\beta)$.

$$AB=c=\frac{2v^2}{g}\frac{\cos\alpha\sin(\alpha-\beta)}{\cos\beta}=\frac{2v^2c^2}{g\,a^2}\sin(\alpha-\beta)\cos\alpha.$$